RAMAKRISHNA MISSION VIDYAMANDIRA

(Residential Autonomous College under University of Calcutta)

B.A./B.SC. FIFTH SEMESTER EXAMINATION, DECEMBER 2013

THIRD YEAR

MATHEMATICS (Honours)

Date : 16/12/2013

Time : 11 am – 3 pm

Paper : V

Full Marks : 100

[Use a separate Answer Book for each group]

Group – A

Answer **any five** questions :

1. a) Let M, N be normal subgroups of a group G such that $M \cap N = \{e\}$ where 'e' is the identity element in G. Show that $xy = yx \, \forall x \in M \,\&\, \forall y \in N$.
   b) Let H be a finite subgroup of group G. If G contains no other subgroup with o(H) many elements, prove that H is normal in G. Deduce that $\{i, (1,2)(3,4), (1,3)(2,4), (1,4)(2,3)\}$ is normal in A_4.
   c) Give an example of a group G such that for each $n \in \mathbb{N}$, $\exists\, x \in G$ with o(x) = n. Justify your answer. [2+5+3]

2. a) Find all the homomorphisms from $\mathbb{Z}_6$ to $\mathbb{Z}_4$.
   b) If G is a group of order 2p, p an odd prime, then prove that either G is commutative or G contains a normal subgroup of order p.
   c) Find all subgroups of $4\mathbb{Z}/64\mathbb{Z}$. [4+3+3]

3. a) Show that S_3 has only 3 homomorphic images.
   b) Let G be a cyclic group of order n. Prove that Aut G is a group of order $\phi(n)$.
   c) Show that there does not exist an onto homomorphism from S_3 onto $\mathbb{Z}_6$. [3+5+2]

4. a) Prove that $\mathbb{Z} \times \mathbb{Z}$ is not cyclic.
   b) Prove that the multiplicative group $\mathbb{C}^*$ of all non zero complex numbers is an internal direct product of the subgroups $\mathbb{R}^+$ and $\{z \in \mathbb{C} : |z| = 1\}$.
   c) Prove that $\mathbb{Z}_m \times \mathbb{Z}_n \cong \mathbb{Z}_{mn}$ iff gcd (m,n) = 1. [2+3+5]

5. a) If G is a group of order p^2, where p is a prime, then prove that G is commutative.
   b) Prove that for each prime p, a finite group G has Sylow p-subgroup.
   c) State and prove Sylow's 2^{nd} theorem. [2+2+6]

6. a) Prove that no group of order 56 is simple.
   b) Let p, q be primes. Prove that no group of order p^2q^2 is simple. [5+5]

7. a) Prove that every finite integral domain is a field.
   b) Show that the identity map is the only field isomorphism from $\mathbb{R}$ onto $\mathbb{R}$.
   c) Let R be a ring with 1. If a is an element of R such that $a^n = 0$ for some positive integer n, prove that 1–a is a unit in R. [3+4+3]

8. a) Prove that every commutative ring with 1 contains a maximal ideal. Verify this theorem for $(\mathbb{Z}, +, \bullet)$.
   b) Let R be an integral domain such that every ideal of R is a prime ideal. Show that R is a field.
   c) Let R be a commutative ring with 1. If A and B are two distinct maximal ideals of R, prove that $AB = A \cap B$. [(3+2)+2+3]

Group – B

Answer **any six** questions :

9. Show that the remainder in approximating f(x) by a interpolating polynomial using distinct interpolating points $x_0, x_1, \ldots, x_n$ is of the form $\dfrac{(x - x_0)(x - x_1)\ldots(x - x_n) f^{n+1}(\xi)}{\lfloor n+1}$ [5]

10. For equally spaced interpolating points $x_i = x_0 + ih,\ i = 0(1)n$;

prove that $\Delta^k y_0 = \sum_{i=0}^{k} (-1)^i \binom{k}{i} y_{k-i}$, where $y_i = f(x_i), i = 0(1)n$. What is inverse intrpolation? [4+1]

11. Define divided difference of order n of a function f. Show that divided difference is a symmetric function of its arguments. [1+4]

12. From the following table, compute $f'(1{\cdot}01)$ and $f''(1{\cdot}01)$, using Stirling's differentiation formulae :

x :	0·96	0·98	1·00	1·02	1·04
f(x) :	0·7825	0·7739	0·7651	0·7563	0·7473

[5]

13. Deduce Simpson one-third rule from Newton – Cote's quadrature formula of closed type. What is the degree of precision for (n+1) points Newton – Cole's formula? [4+1]

14. Discuss Gauss-Seidel iterative method for solving a system of simultaneous linear equations. State the sufficient condition of convergence of the method. [4+1]

15. Deduce Newton-Raphson method for computing a simple root of f(x) = 0. Give the geometrical interpretation of the method. State the order of convergence of the method. [3+1+1]

16. Find y(4·4) by Euler's modified method taking h = 0·2 from the differential equation :

$\frac{dy}{dx} = \frac{2-y^2}{5x}, y = 1$ when $x = 4$. [5]

17. Solve the differential equation : $\frac{dy}{dx} = x^2 + y^2,\ y(0) = 1$

by 4^{th} order Runge-Kutta method from x = 0 to x = 0·2 with step length h = 0·1. [5]

Answer **<u>any four</u>** questions :

18. If a and b are integers, not both zero, then prove that there exist integers u and v such that gcd(a,b) = au+bv. [5]

19. a) If a is prime to b, prove that a+b is to prime to ab. [3]

b) If a is prime to b, then b^2 is prime to a. [2]

20. Show that the general solution of the equation 2x+3y=c, c is a positive integer is x = cu – 3t, y = cv + 2t where $t = 0, \pm 1, \ldots$ and u, v are such tht 2u + 3v = 1.

Further show that for positive integral solution $-\frac{cv}{2} < t < \frac{cu}{3}$. [4+1]

21. a) Prove that the number of primes is infinite. [3]

b) If $2^n - 1$ be a prime, prove that n is a prime, where $n \in \mathbb{N}$. [2]

22. a) Find $\phi(3275)$ and $\mu(292)$ where ϕ and μ have usual meaning. [2]

b) Prove möbius inversion formula. [3]

23. a) State Fundamental theorem of Arithmetic. [1]

b) If p and p+2 be a pair of twin primes, prove that $4(p-1)! + p + 4 \equiv 0 \pmod{p(p+2)}$ [4]

<u>Or</u>

State Wilson's theorem and show that $n = \sum_{d|n} \phi(d)$ (where ϕ has its usual meaning) [4]